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Game-Theoretic Applications of Pentagonal Fuzzy Numbers through New Representation and Defuzzification Schemes

Ganesh Kumar[1,*], Vinod Jangid[1], Govind Shay Sharma[1]

[1] Department of Mathematics, University of Rajasthan, Jaipur-302004, India; ganeshmandha1988@gmail.com; vinodjangid124@gmail.com; gauravsharmaandc@gmail.com.

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Abstract

This research paper presents novel insights into the representation, ranking, and defuzzification of pentagonal fuzzy numbers. The fundamental foundation of this study lies in the development of a robust pentagonal fuzzy number, which is explored through diverse representations that harness the principles of continuity within the membership function. To facilitate practical implementation, a variety of defuzzification methods are meticulously applied, resulting in the transformation of fuzzy data into crisp. The significance of pentagonal fuzzy numbers is further illuminated through their application in the context of game theory, specifically within the domain of matrix games. This strategic analysis explains the pragmatic relevance of pentagonal fuzzy numbers in deciphering complex real-world scenarios and optimizing decision-making processes. Theoretical constructs are bolstered by numerical examples, empirically showcasing the practical applicability of the developed theory.

Keywords: Pentagonal fuzzy number, Ranking, Defuzzification, Two person zero sum game.

1 Introduction

It is commonly believed that all details accessed by the entities are represented in crisp numbers. However, most real-life systems are engaged when the goals and constraints are often vague or unclear. Due to the scarcity of accurate data and details concerning the decision-maker, the fuzzy set theory presented by [28] has a vital role

Corresponding Author: snigdharanip@gift.edu.in

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in the uncertainty theory. [11, 12] analyzed numerous procedures. Type 2 fuzzy sets or interval-valued fuzzy sets designed by [29]. [3, 4] submitted intuitionistic fuzzy sets. [27] evolved fuzzy multi-sets. [24] delivered the concept of neutrosophic fuzzy sets. [5] specified the probability distribution of the objective function in light of parameters expressed as fuzzy triangular numbers. [8] suggested a new technique for ranking fuzzy numbers based on length. [14] supplied the theoretical basis of fuzzy numbers, which was associated with the possibility and Dempster-Shafer theories and conveyed various types of fuzzy number expressions. [9] suggested a novel arithmetic notion and process for conducting arithmetic operations on triangular-fuzzy numbers. [16] depicted fuzzy arithmetic operations to eradicate some of the ambushes of the classical approach. From the perspective of analytical geometry, [25] designated the centroid formulae for fuzzy numbers. [2] offered a defuzzification using the two fuzzy number's distance minimizer. Several modes of ordering trapezoidal fuzzy numbers were conceived by [1]. [19] submitted trapezoidal intuitionistic fuzzy numbers (IF numbers) and revealed some of its operations. Some further arithmetic operators of the trapezoidal fuzzy numbers utilized in [26] suggested an efficient explanation for fuzzy risk investigation. [7] furnished a novel hazard analysis procedure based on a new tool for generalized hierarchies. A novel resolution to a typical fuzzy transportation problem was originated by [15]. [21] analyzed symmetric fuzzy numbers, specified an equivalent definition of convex fuzzy sets, and proposed a technique for creating a symmetric convex fuzzy set. [17] proclaimed that no underlying membership functions were needed and explained many aggregation operators. A further procedure for defuzzifying generalized fuzzy numbers was designed by [22]. [18] presented and approximated many strategies for total ordering in the intuitionistic fuzzy numbers (IFNs) category. [10] suggested ranking fuzzy numbers with widely deviating stretches. [13] presented an alternate technique for estimating additional arithmetic operations of a system using the sigmoidal number under the fuzzy background. [23] involved fuzzy arithmetic operations to equations employed to describe anticipation in multiple applications. The idea of pentagonal-fuzzy number (PFN) was submitted in a broad insight by [20].[6] formulated pentagonal-fuzzy numbers with other expressions to analyze the consequence of optimal strategies for the candidates in real game situations.

This article covers all defuzzification formulas, expressions, and techniques by considering the continuity of membership functions. The solution to the game problem is ultimately detected, and the results are compared to [6]. Subsequently, the present paper provides the updated version of optimal strategies and values of matrix games mentioned in [6].

This article is prepared: Section 2 includes preliminaries and pentagonal-fuzzy numbers with various expressions, and their α cuts are consulted in Section 3. Section 4 contracts with three defuzzification procedures of linear pentagonal fuzzy numbers with symmetry. In Section 5 hierarchy of pentagonal-fuzzy numbers is depicted. Three numerical illustrations and suggested defuzzification approaches have numerically resembled in Section 6. Section 7 finishes with the conclusion of this paper.

2|Preliminaries

In this section, we recall some basic definitions and notations which are useful throughout the paper.

Definition 1. *A fuzzy set $\tilde{A}$ is the collection of ordered pairs $(x, \mu_{\tilde{A}}(x))$ defined on the universal set X and denoted by $\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) | x \in X \, and \, \mu_{\tilde{A}}(x) \in [0,1]\}$, where $\mu_{\tilde{A}} : X \to [0,1]$ is the membership function of $\tilde{A}$It is commonly believed that all details accessed by the entities are represented in crisp numbers. However, most real-life systems are engaged when the goals and constraints are often vague or unclear.*

Definition 2. *An interval valued fuzzy set $\tilde{A}$ defined on the universal set X is represented by $\tilde{A} = \{(x, (\mu_{\tilde{A}^U}(x), \mu_{\tilde{A}^L}(x))) \, | x \in X\}$, where $\mu_{\tilde{A}^U} : X \to [0, \lambda]$ and*

$\mu_{\tilde{A}^L} : X \to [0,\omega]$ *are the upper and lower membership functions with the maximum values* λ *and* ω *respectively.*

Definition 3. *A non linear interval valued triangular fuzzy number* $\tilde{A}_{nLIVFN}$ *is denoted by* $\tilde{A}_{nLIVFN} = [\{(a_1, b, c_1; \lambda), (a, b, c; \omega)\}; \xi_1, \xi_2, \xi_3, \xi_4]$ *where* $0 < \omega \leq \lambda \leq 1$ *and* $a_1 < a < b < c < c_1$. *The upper and lower membership functions are defined as follows:*

$$\mu_{\tilde{A}^U}(x) = \begin{cases} \lambda \left(\frac{x-a_1}{b-a_1}\right)^{\xi_1} & ; a_1 \leq x \leq b \\ \lambda & ; x = b \\ \lambda \left(\frac{c_1-x}{c_1-b}\right)^{\xi_2} & ; b \leq x \leq c_1 \\ 0 & ; else \end{cases} \tag{1}$$

and

$$\mu_{\tilde{A}^L}(x) = \begin{cases} \omega \left(\frac{x-a}{b-a}\right)^{\xi_3} & ; a \leq x \leq b \\ \omega & ; x = b \\ \omega \left(\frac{c-x}{c-b}\right)^{\xi_4} & ; b \leq x \leq c \\ 0 & ; else \end{cases} \tag{2}$$

3|Pentagonal fuzzy number and its different representations

Within this section, we have established distinct categories of pentagonal fuzzy numbers by leveraging the continuity of the membership function—a dimension that was notably absent in the framework presented by [6].

Definition 4. *A pentagonal fuzzy number with* $\mu_{\tilde{A}}(x)$ *as membership function is represented by* $\tilde{A} = (a_1, a_2, a_3, a_4, a_5)$ *and satisfies the following conditions:*

1. $\mu_{\tilde{A}}(x)$ *is continuous in* $[0, 1]$.
2. $\mu_{\tilde{A}}(x)$ *is strictly increasing continuous function on* $[a_1, a_2]$ *and* $[a_2, a_3]$.
3. $\mu_{\tilde{A}}(x)$ *is strictly decreasing continuous function on* $[a_3, a_4]$ *and* $[a_4, a_5]$.

Definition 5. *Linear pentagonal fuzzy number with symmetry* (LS) *is represented by* $\tilde{A}_{LS} = \langle a_1, a_2, a_3, a_4, a_5; k\rangle$. *The revised definition by the Figure 1 with a continuous membership function is given as follows:*

$$\mu_{\tilde{A}_{LS}}(x) = \begin{cases} k\left(\frac{x-a_1}{a_2-a_1}\right) & ; a_1 \leq x \leq a_2 \\ k + (1-k)\left(\frac{x-a_2}{a_3-a_2}\right) & ; a_2 \leq x \leq a_3 \\ 1 & ; x = a_3 \\ k + (1-k)\left(\frac{a_4-x}{a_4-a_3}\right) & ; a_3 \leq x \leq a_4 \\ k\left(\frac{a_5-x}{a_5-a_4}\right) & ; a_4 \leq x \leq a_5 \\ 0 & ; else \end{cases} \tag{3}$$

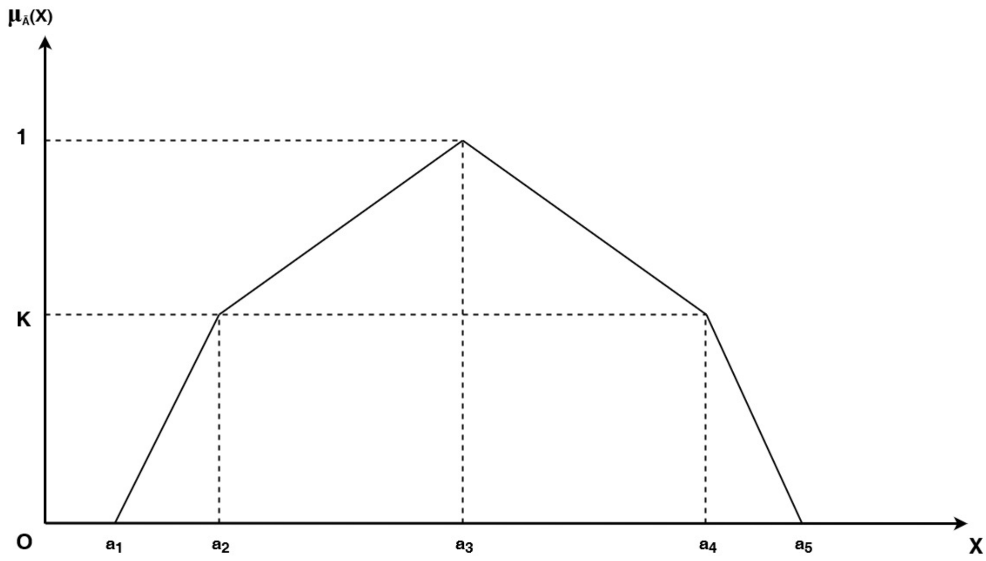

Fig. 1 Linear PFN with symmetry (LS).

Definition 6. *The revised α-cut of linear pentagonal fuzzy number with symmetry is defined as the collection of all $x \in X$ having membership function $\mu_{\tilde{A}_{LS}}(x)$ as greater than or equal to α i.e. $A_\alpha = \{x \in X | \mu_{\tilde{A}_{LS}}(x) \geq \alpha\}$ is given by*

$$A_\alpha = \begin{cases} A_{1L}(\alpha) = a_1 + \frac{\alpha}{k}(a_2 - a_1) & ;\alpha \in [0,k] \\ A_{2L}(\alpha) = a_2 + \frac{\alpha - k}{1-k}(a_3 - a_2) & ;\alpha \in [k,1] \\ A_{2R}(\alpha) = a_4 - \frac{\alpha - k}{1-k}(a_4 - a_3) & ;\alpha \in [k,1] \\ A_{1R}(\alpha) = a_5 - \frac{\alpha}{k}(a_5 - a_4) & ;\alpha \in [0,k] \end{cases} \tag{4}$$

Here $A_{1L}(\alpha), A_{2L}(\alpha)$ are the increasing functions of α and $A_{1R}(\alpha), A_{2R}(\alpha)$ are the decreasing functions of α.

Definition 7. *Non linear pentagonal fuzzy number with symmetry (NS) is represented by $\tilde{A}_{NS} = (a_1, a_2, a_3, a_4, a_5; k)_{(\xi_1, \xi_2; \zeta_1, \zeta_2)}$. The revised definition by the Figure 2 with a continuous membership function is given as follows:*

$$\mu_{\tilde{A}_{NS}}(x) = \begin{cases} k\left(\frac{x-a_1}{a_2-a_1}\right)^{\xi_1} & ;a_1 \leq x \leq a_2 \\ k + (1-k)\left(\frac{x-a_2}{a_3-a_2}\right)^{\xi_2} & ;a_2 \leq x \leq a_3 \\ 1 & ;x = a_3 \\ k + (1-k)\left(\frac{a_4-x}{a_4-a_3}\right)^{\zeta_1} & ;a_3 \leq x \leq a_4 \\ k\left(\frac{a_5-x}{a_5-a_4}\right)^{\zeta_2} & ;a_4 \leq x \leq a_5 \\ 0 & ;else \end{cases} \tag{5}$$

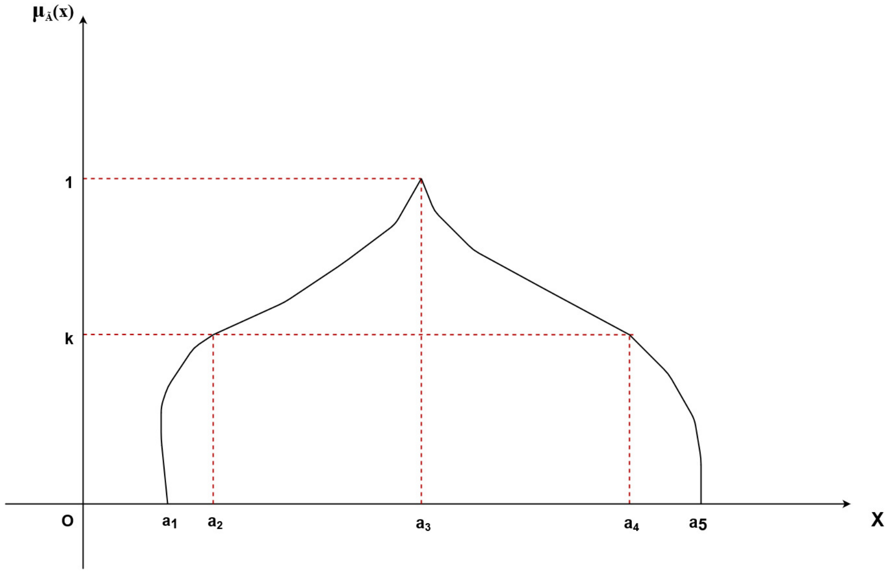

Fig. 2 Non Linear PFN with Symmetry $\tilde{A}_{NS}$.

Definition 8. *The revised* α*-cut of non linear pentagonal fuzzy number with symmetry is defined as the collection of all* $x \in X$ *with membership function* $\mu_{\tilde{A}_{NS}}(x)$ *as greater than or equal to* α *i.e.* $A_\alpha = \{x \in X | \mu_{\tilde{A}_{NS}}(x) \geq \alpha\}$ *is*

$$A_\alpha = \begin{cases} A_{1L}(\alpha) = a_1 + \left(\frac{\alpha}{k}\right)^{\frac{1}{\xi_1}} (a_2 - a_1) & ; \alpha \in [0, k] \\ A_{2L}(\alpha) = a_2 + \left(\frac{\alpha-k}{1-k}\right)^{\frac{1}{\xi_2}} (a_3 - a_2) & ; \alpha \in [k, 1] \\ A_{2R}(\alpha) = a_4 - \left(\frac{\alpha-k}{1-k}\right)^{\frac{1}{\zeta_1}} (a_4 - a_3) & ; \alpha \in [k, 1] \\ A_{1R}(\alpha) = a_5 - \left(\frac{\alpha}{k}\right)^{\frac{1}{\zeta_2}} (a_5 - a_4) & ; \alpha \in [0, k] \end{cases} \quad (6)$$

Here $A_{1L}(\alpha), A_{2L}(\alpha)$ *are the increasing functions of* α *and* $A_{1R}(\alpha), A_{2R}(\alpha)$ *are the decreasing functions of* α.

Definition 9. *The revised definition of linear interval valued pentagonal fuzzy number with symmetry (LIPS)* $\tilde{A}_{LIPS} = \langle (a_1, a_2, c, a_4, a_5; k, p), (b_1, b_2, c, b_4, b_5; w, q) \rangle$ *is* *defined by the Figure 3 with upper and lower membership functions as follows:*

$$\mu_{\tilde{A}^U_{LIPS}}(x) = \begin{cases} p\left(\frac{x-a_1}{a_2-a_1}\right) & ; a_1 \le x \le a_2 \\ p+(k-p)\left(\frac{x-a_2}{c-a_2}\right) & ; a_2 \le x \le c \\ k & ; x = c \\ p+(k-p)\left(\frac{a_4-x}{a_4-c}\right) & ; c \le x \le a_4 \\ p\left(\frac{a_5-x}{a_5-a_4}\right) & ; a_4 \le x \le a_5 \\ 0 & ; else \end{cases} \tag{7}$$

and

$$\mu_{\tilde{A}^L_{LIPS}}(x) = \begin{cases} q\left(\frac{x-b_1}{b_2-b_1}\right) & ; b_1 \le x \le b_2 \\ q+(w-q)\left(\frac{x-b_2}{c-b_2}\right) & ; b_2 \le x \le c \\ w & ; x = c \\ q+(w-q)\left(\frac{b_4-x}{b_4-c}\right) & ; c \le x \le b_4 \\ q\left(\frac{b_5-x}{b_5-b_4}\right) & ; b_4 \le x \le b_5 \\ 0 & ; else \end{cases} \tag{8}$$

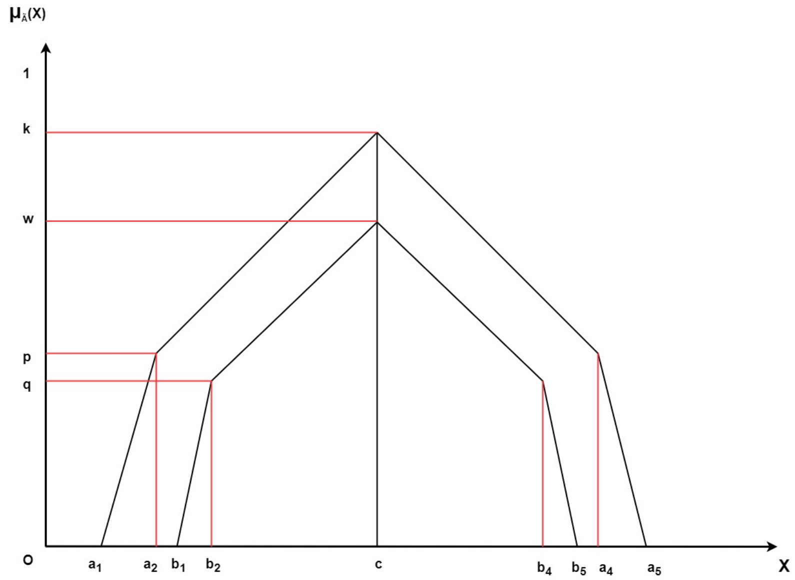

Fig. 3. Linear IVPFN with symmetry ($LIPS$).

Definition 10. *The revised α-cut of linear interval valued pentagonal fuzzy number with symmetry is defined as* $\tilde{A}_{LIPS\alpha} = \bigcup_{\alpha_2}\left(\tilde{A}^U_{LIPS}\right)_{\alpha_2} \ominus \bigcup_{\alpha_1}\left(\tilde{A}^L_{LIPS}\right)_{\alpha_1}$ *for* $\alpha_1 \in [0, w]$;

$\alpha_2 \in [0, k]$.
Where

$$\left(\tilde{A}^U_{LIPS}\right)_{\alpha_2} = \begin{cases} A^U{}_{1L}(\alpha_2) = a_1 + \frac{\alpha_2}{p}(a_2 - a_1) & ; \alpha_2 \in [0, p] \\ A^U{}_{2L}(\alpha_2) = a_2 + \frac{\alpha_2 - p}{k - p}(c - a_2) & ; \alpha_2 \in [p, k] \\ A^U{}_{2R}(\alpha_2) = a_4 - \frac{\alpha_2 - p}{k - p}(a_4 - c) & ; \alpha_2 \in [p, k] \\ A^U{}_{1R}(\alpha_2) = a_5 - \frac{\alpha_2}{p}(a_5 - a_4) & ; \alpha_2 \in [0, p] \end{cases} \tag{9}$$

and

$$\left(\tilde{A}^L_{LIPS}\right)_{\alpha_1} = \begin{cases} A^L{}_{1L}(\alpha_1) = b_1 + \frac{\alpha_1}{q}(b_2 - b_1) & ; \alpha_1 \in [0, q] \\ A^L{}_{2L}(\alpha_1) = b_2 + \frac{\alpha_1 - q}{w - q}(c - b_2) & ; \alpha_1 \in [q, w] \\ A^L{}_{2R}(\alpha_1) = b_4 - \frac{\alpha_1 - q}{w - q}(b_4 - c) & ; \alpha_1 \in [q, w] \\ A^L{}_{1R}(\alpha_1) = b_5 - \frac{\alpha_1}{q}(b_5 - b_4) & ; \alpha_1 \in [0, q] \end{cases} \tag{10}$$

Here $A^U{}_{1L}(\alpha_2)$, $A^U{}_{2L}(\alpha_2)$, $A^L{}_{1L}(\alpha_1)$, $A^L{}_{2L}(\alpha_1)$ *are the increasing functions of* α_2 *and* α_1 *correspondingly and* $A^U{}_{1R}(\alpha_2)$, $A^U{}_{2R}(\alpha_2)$, $A^L{}_{1R}(\alpha_1)$, $A^L{}_{2R}(\alpha_1)$ *are the decreasing functions of* α_2 *and* α_1 *correspondingly.*

Definition 11. *The membership functions of linear interval valued pentagonal fuzzy number with asymmetry* $(LIPAS)$ $\tilde{A}_{LIPAS} = \langle (a_1, a_2, c, a_4, a_5; k, p, r), (b_1, b_2, c, b_4, b_5; w, q, s) \rangle$ *are defined by the Figure 4 with upper and lower membership functions as follows:*

$$\mu_{\tilde{A}^U_{LIPAS}}(x) = \begin{cases} p\left(\frac{x - a_1}{a_2 - a_1}\right) & ; a_1 \leq x \leq a_2 \\ p + (k - p)\left(\frac{x - a_2}{c - a_2}\right) & ; a_2 \leq x \leq c \\ k & ; x = c \\ r + (k - r)\left(\frac{a_4 - x}{a_4 - c}\right) & ; c \leq x \leq a_4 \\ r\left(\frac{a_5 - x}{a_5 - a_4}\right) & ; a_4 \leq x \leq a_5 \\ 0 & ; else \end{cases} \tag{11}$$

and

$$\mu_{\tilde{A}^L_{LIPAS}}(x) = \begin{cases} q\left(\frac{x - b_1}{b_2 - b_1}\right) & ; b_1 \leq x \leq b_2 \\ q + (w - q)\left(\frac{x - b_2}{c - b_2}\right) & ; b_2 \leq x \leq c \\ w & ; x = c \\ s + (w - s)\left(\frac{b_4 - x}{b_4 - c}\right) & ; c \leq x \leq b_4 \\ s\left(\frac{b_5 - x}{b_5 - b_4}\right) & ; b_4 \leq x \leq b_5 \\ 0 & ; else \end{cases} \tag{12}$$

Definition 12. *The revised* α*-cut of linear interval valued pentagonal fuzzy number with asymmetry is defined as* $A_{LIPAS\alpha} = \bigcup_{\alpha_2}\left(\tilde{A}^U_{LIPAS}\right)_{\alpha_2} \ominus \bigcup_{\alpha_1}\left(\tilde{A}^L_{LIPAS}\right)_{\alpha_1}$ *for* $\alpha_1 \in [0, w]$*;* $\alpha_2 \in [0, k]$.

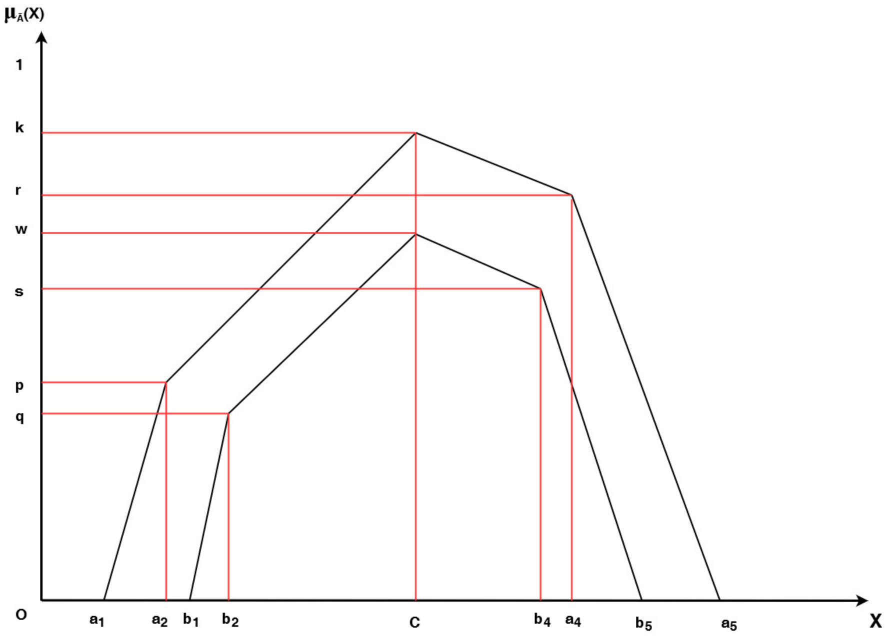

Fig. 4. Linear IVPFN with asymmetry ($LIPAS$).

Where

$$\left(\tilde{A}^{U}_{LIPAS}\right)_{\alpha_2}=\begin{cases} A^{U}{}_{1L}(\alpha_2)=a_1+\frac{\alpha_2}{p}\left(a_2-a_1\right) & ;\alpha_2\in[0,p] \\ A^{U}{}_{2L}(\alpha_2)=a_2+\frac{\alpha_2-p}{k-p}\left(c-a_2\right) & ;\alpha_2\in[p,k] \\ A^{U}{}_{2R}(\alpha_2)=a_4-\frac{\alpha_2-r}{k-r}\left(a_4-c\right) & ;\alpha_2\in[r,k] \\ A^{U}{}_{1R}(\alpha_2)=a_5-\frac{\alpha_2}{r}\left(a_5-a_4\right) & ;\alpha_2\in[0,r] \end{cases} \tag{13}$$

and

$$\left(\tilde{A}^{L}_{LIPAS}\right)_{\alpha_1}=\begin{cases} A^{L}{}_{1L}(\alpha_1)=b_1+\frac{\alpha_1}{q}\left(b_2-b_1\right) & ;\alpha_1\in[0,q] \\ A^{L}{}_{2L}(\alpha_1)=b_2+\frac{\alpha_1-q}{w-q}\left(c-b_2\right) & ;\alpha_1\in[q,w] \\ A^{L}{}_{2R}(\alpha_1)=b_4-\frac{\alpha_1-s}{w-s}\left(b_4-c\right) & ;\alpha_1\in[s,w] \\ A^{L}{}_{1R}(\alpha_1)=b_5-\frac{\alpha_1}{s}\left(b_5-b_4\right) & ;\alpha_1\in[0,s] \end{cases} \tag{14}$$

Here $A^{U}{}_{1L}(\alpha_2)$, $A^{U}{}_{2L}(\alpha_2)$, $A^{L}{}_{1L}(\alpha_1)$, $A^{L}{}_{2L}(\alpha_1)$ *are the increasing functions of* α_2 *and* α_1 *correspondingly and* $A^{U}{}_{1R}(\alpha_2)$, $A^{U}{}_{2R}(\alpha_2)$, $A^{L}{}_{1R}(\alpha_1)$, $A^{L}{}_{2R}(\alpha_1)$ *are the decreasing functions of* α_2 *and* α_1 *correspondingly.*

Definition 13. *The membership functions of non linear interval valued pentagonal fuzzy number with symmetry* ($NIPS$) $\tilde{A}_{NIPS}=\left\langle (a_1,a_2,c,a_4,a_5;k,p)_{(\xi_1,\xi_2;\zeta_1,\zeta_2)},(b_1,b_2,c,b_4,b_5;w,q)_{(\xi_1,\xi_2;\zeta_1,\zeta_2)}\right\rangle$ *are defined by the Figures 5 to 8 for different parameters* $\xi_1,\xi_2,\zeta_1,\zeta_2$ *with revised continuous upper and lower membership functions as follows:*

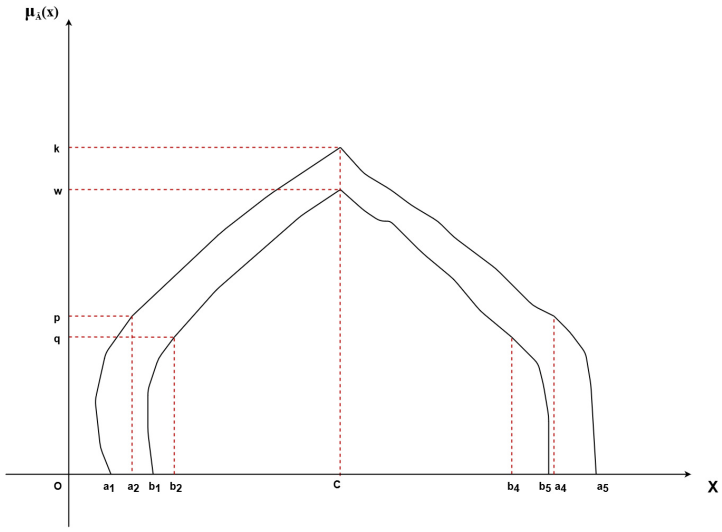

Fig. 5. $\tilde{A}_{NIPS}$ for $\xi_1, \xi_2, \zeta_1, \zeta_2 > 1$.

$$\mu_{\tilde{A}^U_{NIPS}}(x) = \begin{cases} p\left(\frac{x-a_1}{a_2-a_1}\right)^{\xi_1} & ; a_1 \leq x \leq a_2 \\ p+(k-p)\left(\frac{x-a_2}{c-a_2}\right)^{\xi_2} & ; a_2 \leq x \leq c \\ k & ; x = c \\ p+(k-p)\left(\frac{a_4-x}{a_4-c}\right)^{\zeta_1} & ; c \leq x \leq a_4 \\ p\left(\frac{a_5-x}{a_5-a_4}\right)^{\zeta_2} & ; a_4 \leq x \leq a_5 \\ 0 & ; else \end{cases} \tag{15}$$

and

$$\mu_{\tilde{A}^L_{NIPS}}(x) = \begin{cases} q\left(\frac{x-b_1}{b_2-b_1}\right)^{\xi_1} & ; b_1 \leq x \leq b_2 \\ q+(w-q)\left(\frac{x-b_2}{c-b_2}\right)^{\xi_2} & ; b_2 \leq x \leq c \\ w & ; x = c \\ q+(w-q)\left(\frac{b_4-x}{b_4-c}\right)^{\zeta_1} & ; c \leq x \leq b_4 \\ q\left(\frac{b_5-x}{b_5-b_4}\right)^{\zeta_2} & ; b_4 \leq x \leq b_5 \\ 0 & ; else \end{cases} \tag{16}$$

Definition 14. *The revised α-cut of non linear interval valued pentagonal fuzzy number with symmetry is defined as follow:* $A_{NIPS\alpha} = \bigcup_{\alpha_2}\left(\tilde{A}^U_{NIPS}\right)_{\alpha_2} \ominus \bigcup_{\alpha_1}\left(\tilde{A}^L_{NIPS}\right)_{\alpha_1}$ *for* $\alpha_1 \in [0, w]$; $\alpha_2 \in [0, k]$.

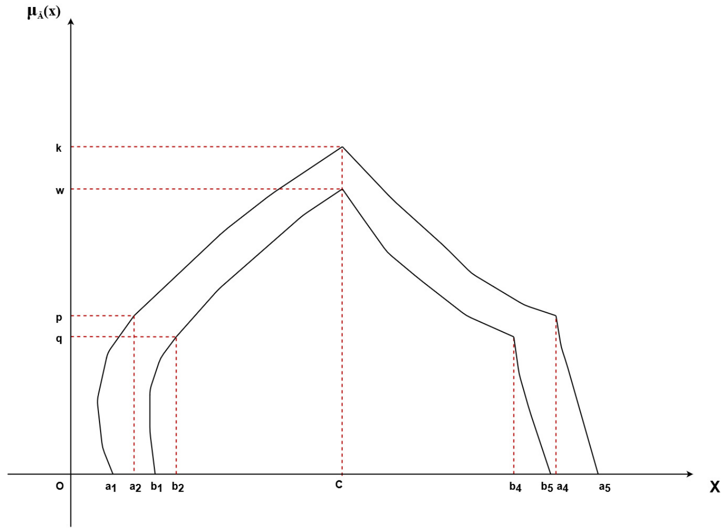

Fig. 6. $\tilde{A}_{NIPS}$ for $\xi_1, \xi_2 > 1; \zeta_1, \zeta_2 < 1$.

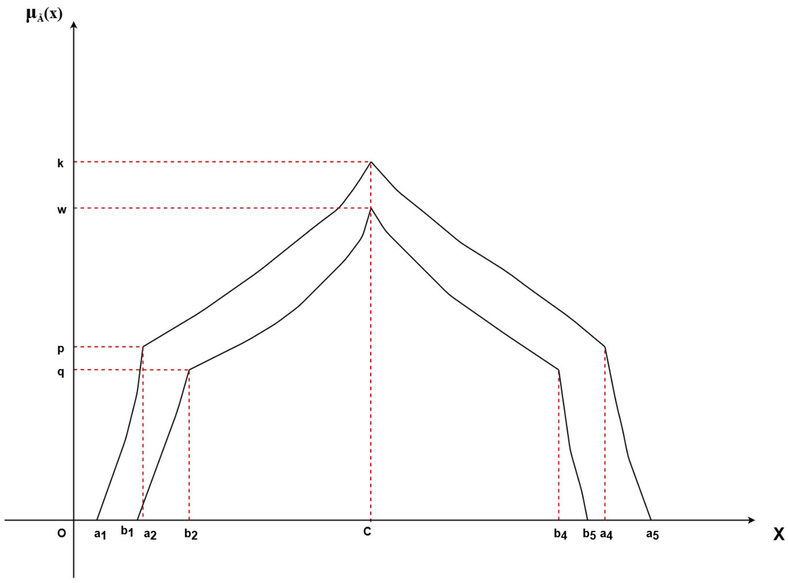

Fig. 7. $\tilde{A}_{NIPS}$ for $\xi_1, \xi_2, \zeta_1, \zeta_2 < 1$.

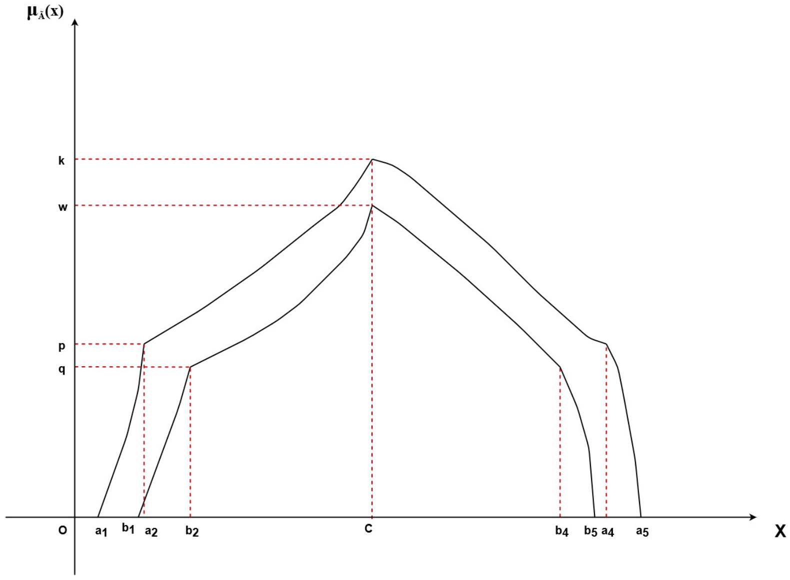

Fig. 8. $\tilde{A}_{NIPS}$ for $\xi_1, \xi_2 < 1; \zeta_1, \zeta_2 > 1$.

Where

$$\left(\tilde{A}^{U}_{NIPS}\right)_{\alpha_2} = \begin{cases} A^{U}{}_{1L}(\alpha_2) = a_1 + \left(\frac{\alpha_2}{p}\right)^{\frac{1}{\xi_1}} (a_2 - a_1) & ; \alpha_2 \in [0, p] \\ A^{U}{}_{2L}(\alpha_2) = a_2 + \left(\frac{\alpha_2 - p}{k - p}\right)^{\frac{1}{\xi_2}} (c - a_2) & ; \alpha_2 \in [p, k] \\ A^{U}{}_{2R}(\alpha_2) = a_4 - \left(\frac{\alpha_2 - p}{k - p}\right)^{\frac{1}{\zeta_1}} (a_4 - c) & ; \alpha_2 \in [p, k] \\ A^{U}{}_{1R}(\alpha_2) = a_5 - \left(\frac{\alpha_2}{p}\right)^{\frac{1}{\zeta_2}} (a_5 - a_4) & ; \alpha_2 \in [0, p] \end{cases} \tag{17}$$

and

$$\left(\tilde{A}^{L}_{NIPS}\right)_{\alpha_1} = \begin{cases} A^{L}{}_{1L}(\alpha_1) = b_1 + \left(\frac{\alpha_1}{q}\right)^{\frac{1}{\xi_1}} (b_2 - b_1) & ; \alpha_1 \in [0, q] \\ A^{L}{}_{2L}(\alpha_1) = b_2 + \left(\frac{\alpha_1 - q}{w - q}\right)^{\frac{1}{\xi_2}} (c - b_2) & ; \alpha_1 \in [q, w] \\ A^{L}{}_{2R}(\alpha_1) = b_4 - \left(\frac{\alpha_1 - q}{w - q}\right)^{\frac{1}{\zeta_1}} (b_4 - c) & ; \alpha_1 \in [q, w] \\ A^{L}{}_{1R}(\alpha_1) = b_5 - \left(\frac{\alpha_1}{q}\right)^{\frac{1}{\zeta_2}} (b_5 - b_4) & ; \alpha_1 \in [0, q] \end{cases} \tag{18}$$

Here $A^{U}{}_{1L}(\alpha_2)$, $A^{U}{}_{2L}(\alpha_2)$, $A^{L}{}_{1L}(\alpha_1)$, $A^{L}{}_{2L}(\alpha_1)$ *are the increasing functions of* α_2 *and* α_1 *correspondingly and* $A^{U}{}_{1R}(\alpha_2)$, $A^{U}{}_{2R}(\alpha_2)$, $A^{L}{}_{1R}(\alpha_1)$, $A^{L}{}_{2R}(\alpha_1)$ *are the decreasing functions of* α_2 *and* α_1 *correspondingly.*

Definition 15. *The membership functions of non linear interval valued pentagonal fuzzy number with asymmetry* $(NIPAS)$ $\tilde{A}_{NIPAS} =$

$\left\langle (a_1, a_2, c, a_4, a_5; k, p, s)_{(\xi_1,\xi_2;\zeta_1,\zeta_2)}, (b_1, b_2, c, b_4, b_5; w, q, t)_{(\xi_1,\xi_2;\zeta_1,\zeta_2)} \right\rangle$ *are defined by the Figures 9 to 12 for different parameters* $\xi_1, \xi_2, \zeta_1, \zeta_2$ *having continuous upper and lower membership functions as follows:*

$$\mu_{\tilde{A}^U_{NIPAS}}(x) = \begin{cases} p\left(\frac{x-a_1}{a_2-a_1}\right)^{\xi_1} & ; a_1 \le x \le a_2 \\ p + (k-p)\left(\frac{x-a_2}{c-a_2}\right)^{\xi_2} & ; a_2 \le x \le c \\ k & ; x = c \\ s + (k-s)\left(\frac{a_4-x}{a_4-c}\right)^{\zeta_1} & ; c \le x \le a_4 \\ s\left(\frac{a_5-x}{a_5-a_4}\right)^{\zeta_2} & ; a_4 \le x \le a_5 \\ 0 & ; else \end{cases} \tag{19}$$

and

$$\mu_{\tilde{A}^L_{NIPAS}}(x) = \begin{cases} q\left(\frac{x-b_1}{b_2-b_1}\right)^{\xi_1} & ; b_1 \le x \le b_2 \\ q + (w-q)\left(\frac{x-b_2}{c-b_2}\right)^{\xi_2} & ; b_2 \le x \le c \\ w & ; x = c \\ t + (w-t)\left(\frac{b_4-x}{b_4-c}\right)^{\zeta_1} & ; c \le x \le b_4 \\ t\left(\frac{b_5-x}{b_5-b_4}\right)^{\zeta_2} & ; b_4 \le x \le b_5 \\ 0 & ; else \end{cases} \tag{20}$$

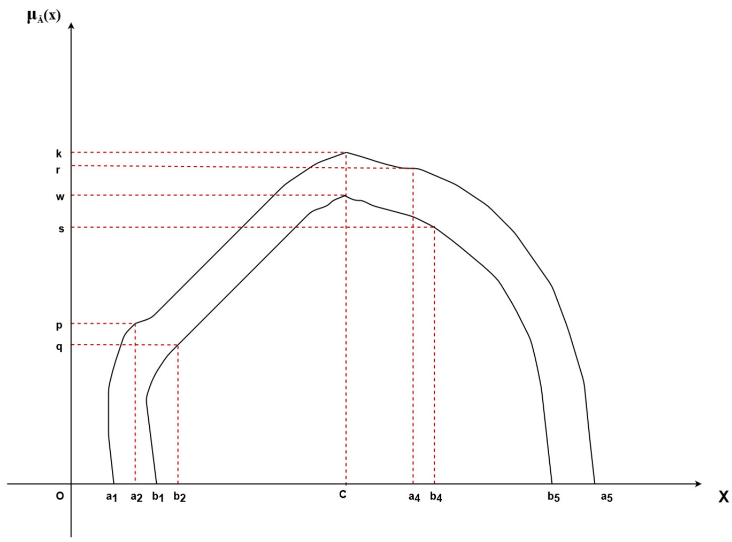

Fig. 9. $\tilde{A}_{NIPAS}$ for $\xi_1, \xi_2, \zeta_1, \zeta_2 > 1$.

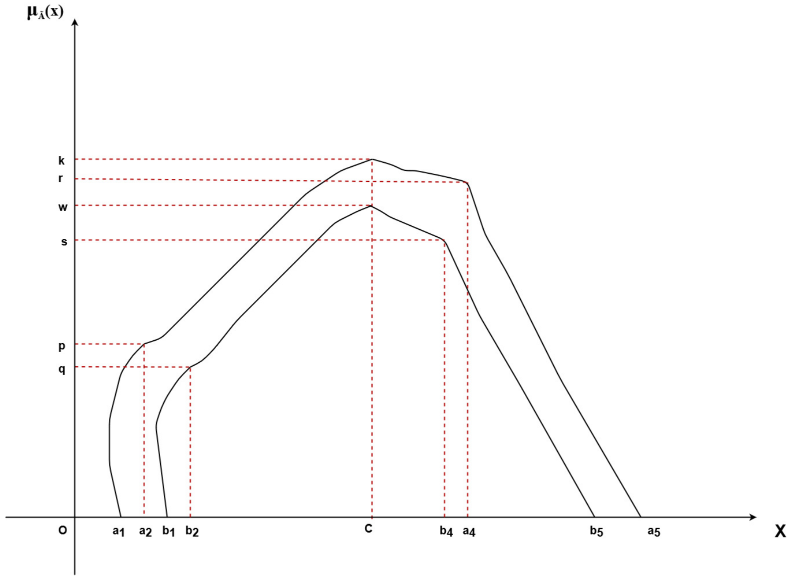

Fig. 10. $\tilde{A}_{NIPAS}$ for $\xi_1, \xi_2 > 1; \zeta_1, \zeta_2 < 1$.

Definition 16. *The revised α-cut of non linear interval valued pentagonal fuzzy number with asymmetry defined as* $A_{NIPAS\alpha} = \bigcup_{\alpha_2} \left(\tilde{A}^U_{NIPAS}\right)_{\alpha_2} \ominus \bigcup_{\alpha_1} \left(\tilde{A}^L_{NIPAS}\right)_{\alpha_1}$ *for* $\alpha_1 \in [0, w]$*;* $\alpha_2 \in [0, k]$.
Where

$$\left(\tilde{A}^U_{NIPAS}\right)_{\alpha_2} = \begin{cases} A^U{}_{1L}(\alpha_2) = a_1 + \left(\frac{\alpha_2}{p}\right)^{\frac{1}{\xi_1}} (a_2 - a_1) & ; \alpha_2 \in [0, p] \\ A^U{}_{2L}(\alpha_2) = a_2 + \left(\frac{\alpha_2 - p}{k - p}\right)^{\frac{1}{\xi_2}} (c - a_2) & ; \alpha_2 \in [p, k] \\ A^U{}_{2R}(\alpha_2) = a_4 - \left(\frac{\alpha_2 - s}{k - s}\right)^{\frac{1}{\zeta_1}} (a_4 - c) & ; \alpha_2 \in [s, k] \\ A^U{}_{1R}(\alpha_2) = a_5 - \left(\frac{\alpha_2}{s}\right)^{\frac{1}{\zeta_2}} (a_5 - a_4) & ; \alpha_2 \in [0, s] \end{cases} \tag{21}$$

and

$$\left(\tilde{A}^L_{NIPAS}\right)_{\alpha_1} = \begin{cases} A^L{}_{1L}(\alpha_1) = b_1 + \left(\frac{\alpha_1}{q}\right)^{\frac{1}{\xi_1}} (b_2 - b_1) & ; \alpha_1 \in [0, q] \\ A^L{}_{2L}(\alpha_1) = b_2 + \left(\frac{\alpha_1 - q}{w - q}\right)^{\frac{1}{\xi_2}} (c - b_2) & ; \alpha_1 \in [q, w] \\ A^L{}_{2R}(\alpha_1) = b_4 - \left(\frac{\alpha_1 - t}{w - t}\right)^{\frac{1}{\zeta_1}} (b_4 - c) & ; \alpha_1 \in [t, w] \\ A^L{}_{1R}(\alpha_1) = b_5 - \left(\frac{\alpha_1}{t}\right)^{\frac{1}{\zeta_2}} (b_5 - b_4) & ; \alpha_1 \in [0, t] \end{cases} \tag{22}$$

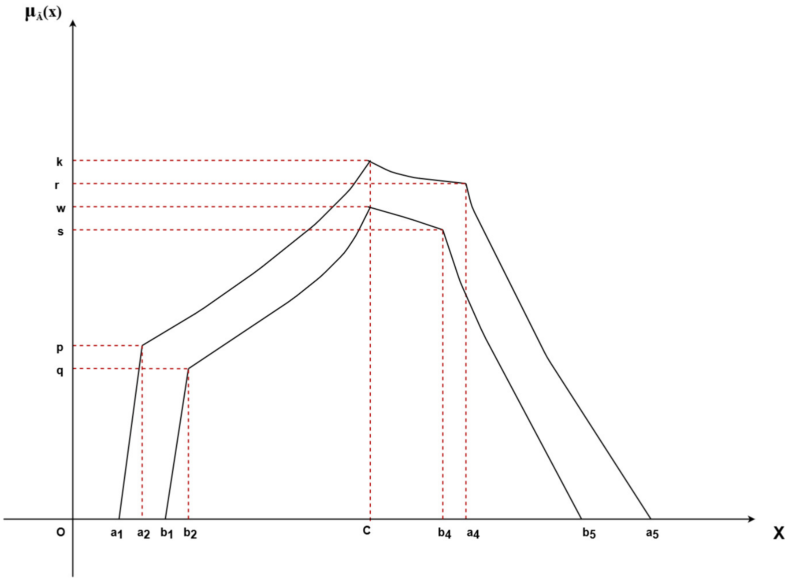

Fig. 11. $\tilde{A}_{NIPAS}$ for $\xi_1, \xi_2, \zeta_1, \zeta_2 < 1$.

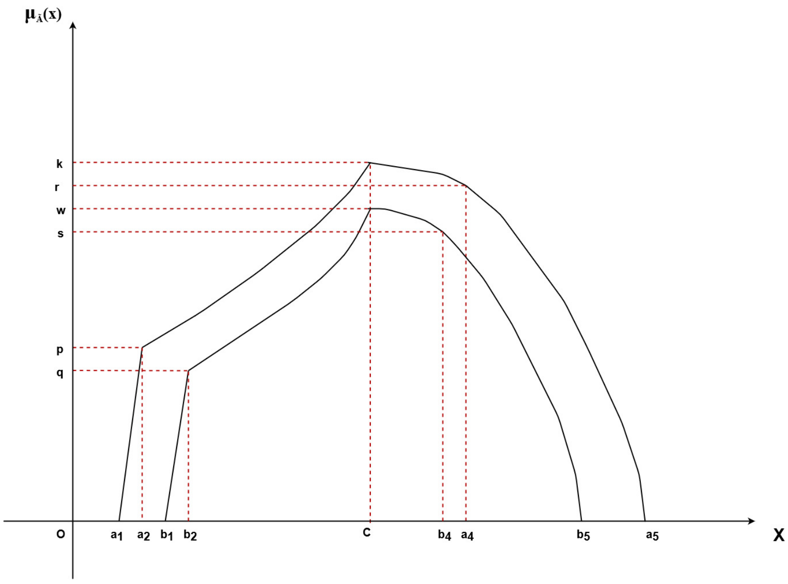

Fig. 12. $\tilde{A}_{NIPAS}$ for $\xi_1, \xi_2 < 1; \zeta_1, \zeta_2 > 1$.

Here $A^U{}_{1L}(\alpha_2)$, $A^U{}_{2L}(\alpha_2)$, $A^L{}_{1L}(\alpha_1)$, $A^L{}_{2L}(\alpha_1)$ are the increasing functions of α_2 and α_1 correspondingly and $A^U{}_{1R}(\alpha_2)$, $A^U{}_{2R}(\alpha_2)$, $A^L{}_{1R}(\alpha_1)$, $A^L{}_{2R}(\alpha_1)$ are the decreasing functions of α_2 and α_1 correspondingly.

4|Defuzzification

The defuzzification method converts the fuzzy inference result into an appropriate exact value. It plays a critical role in a fuzzy environment. For fuzzy logic, several defuzzification techniques are available, such as the center of the area (centroid), greatest of maxima, smallest of maxima, mean of maxima, the bisector of area, graded mean integral value, and so on. We investigate three methods to defuzzify the linear pentagonal fuzzy number with symmetry (LS) based on different approaches. Our results are compared with [6].

4.1|Centroid Method

In this section we have proposed a method for the defuzzification of non linear pentagonal fuzzy number with symmetry as follows:

$$\begin{aligned} R &= \frac{\int_{a_1}^{a_2} x\mu_{\tilde{A}_{NS}}(x)dx+\int_{a_2}^{a_3} x\mu_{\tilde{A}_{NS}}(x)dx+\int_{a_3}^{a_4} x\mu_{\tilde{A}_{NS}}(x)dx+\int_{a_4}^{a_5} x\mu_{\tilde{A}_{NS}}(x)dx}{\int_{a_1}^{a_2} \mu_{\tilde{A}_{NS}}(x)dx+\int_{a_2}^{a_3} \mu_{\tilde{A}_{NS}}(x)dx+\int_{a_3}^{a_4} \mu_{\tilde{A}_{NS}}(x)dx+\int_{a_4}^{a_5} \mu_{\tilde{A}_{NS}}(x)dx} \\ &= \frac{P}{Q} \end{aligned} \quad (23)$$

From *Eqs. (5)* and *(23)* we obtain

$$P = \int_{a_1}^{a_2} x\left[k\left(\frac{x-a_1}{a_2-a_1}\right)^{\xi_1}\right]dx + \int_{a_2}^{a_3} x\left[k+(1-k)\left(\frac{x-a_2}{a_3-a_2}\right)^{\xi_2}\right]dx$$
$$+\int_{a_3}^{a_4} x\left[k+(1-k)\left(\frac{a_4-x}{a_4-a_3}\right)^{\zeta_1}\right]dx + \int_{a_4}^{a_5} x\left[k\left(\frac{a_5-x}{a_5-a_4}\right)^{\zeta_2}\right]dx$$

and

$$Q = \int_{a_1}^{a_2} \left[k\left(\frac{x-a_1}{a_2-a_1}\right)^{\xi_1}\right]dx + \int_{a_2}^{a_3} \left[k+(1-k)\left(\frac{x-a_2}{a_3-a_2}\right)^{\xi_2}\right]dx$$
$$+\int_{a_3}^{a_4} \left[k+(1-k)\left(\frac{a_4-x}{a_4-a_3}\right)^{\zeta_1}\right]dx + \int_{a_4}^{a_5} \left[k\left(\frac{a_5-x}{a_5-a_4}\right)^{\zeta_2}\right]dx$$

On computing we have

$$\begin{aligned} P = &\frac{ka_2(a_2-a_1)}{(\xi_1+1)} - \frac{k(a_2-a_1)^2}{(\xi_1+1)(\xi_1+2)} + \frac{k\left({a_3}^2-{a_2}^2\right)}{2} + \frac{(1-k)a_3(a_3-a_2)}{(\xi_2+1)} \\ &- \frac{(1-k)(a_3-a_2)^2}{(\xi_2+1)(\xi_2+2)} + \frac{k\left({a_4}^2-{a_3}^2\right)}{2} + \frac{(1-k)a_3(a_4-a_3)}{(\zeta_1+1)} \\ &+ \frac{(1-k)(a_4-a_3)^2}{(\zeta_1+1)(\zeta_1+2)} + \frac{ka_4(a_5-a_4)}{(\zeta_2+1)} + \frac{k(a_5-a_4)^2}{(\zeta_2+1)(\zeta_2+2)} \end{aligned} \quad (24)$$

and

$$Q = \frac{k(a_2-a_1)}{(\xi_1+1)} + k(a_3-a_2) + \frac{(1-k)(a_3-a_2)}{(\xi_2+1)} + k(a_4-a_3) + \frac{(1-k)(a_4-a_3)}{(\zeta_1+1)} + \frac{k(a_5-a_4)}{(\zeta_2+1)} \tag{25}$$

If we set $\xi_1 = 1$, $\xi_2 = 1$, $\zeta_1 = 1$ and $\zeta_2 = 1$ then the proposed defuzzification technique applies for the linear pentagonal fuzzy number with symmetry as follows

$$R = \frac{L+M}{N} \tag{26}$$

where

$$L = \frac{ka_2(a_2-a_1)}{2} - \frac{k(a_2-a_1)^2}{6} + \frac{k\left(a_3{}^2-a_2{}^2\right)}{2} + \frac{(1-k)a_3(a_3-a_2)}{2} - \frac{(1-k)(a_3-a_2)^2}{6} \tag{27}$$

$$M = \frac{k\left(a_4{}^2-a_3{}^2\right)}{2} + \frac{(1-k)a_3(a_4-a_3)}{2} + \frac{(1-k)(a_4-a_3)^2}{6} + \frac{ka_4(a_5-a_4)}{2} + \frac{k(a_5-a_4)^2}{6} \tag{28}$$

and

$$N = \frac{k(a_2-a_1)}{2} + k(a_3-a_2) + \frac{(1-k)(a_3-a_2)}{2} + k(a_4-a_3) + \frac{(1-k)(a_4-a_3)}{2} + \frac{k(a_5-a_4)}{2} \tag{29}$$

For $k = 1$ we have

$$R' = \frac{\left(a_4{}^2 + a_5{}^2 + a_4a_5 - a_1{}^2 - a_2{}^2 - a_1a_2\right)}{3(a_4+a_5-a_1-a_2)} \tag{30}$$

4.2|Defuzzification using α-cut

In this section, we propose a method to compute the defuzzification of non-linear pentagonal fuzzy numbers with symmetry as follows:
Since the left and right α-cuts of a non linear pentagonal fuzzy number with symmetry $\tilde{A}_{NS} = \left\langle (a_1, a_2, a_3, a_4, a_5; k)_{(\xi_1,\xi_2;\zeta_1,\zeta_2)} \right\rangle$ are

$$L^{-1}(\alpha) = a_1 + \left(\frac{\alpha}{k}\right)^{\frac{1}{\xi_1}} (a_2 - a_1); \alpha \in [0, k],$$

$$R^{-1}(\alpha) = a_5 - \left(\frac{\alpha}{k}\right)^{\frac{1}{\zeta_2}} (a_5 - a_4); \alpha \in [0, k],$$

$$L^{-1}(\alpha) = a_2 + \left(\frac{\alpha-k}{1-k}\right)^{\frac{1}{\xi_2}} (a_3 - a_2); \alpha \in [k, 1],$$

$$R^{-1}(\alpha) = a_4 - \left(\frac{\alpha-k}{1-k}\right)^{\frac{1}{\zeta_1}} (a_4 - a_3); \alpha \in [k, 1].$$

Then the mean of α- cut of $\tilde{A}_{NS}$ is

$$\hat{A} = \int_{\alpha=0}^{1} \frac{\left(L^{-1}(\alpha) + R^{-1}(\alpha)\right)}{2} d\alpha$$
$$= \int_{\alpha=0}^{k} \frac{\left(L^{-1}(\alpha) + R^{-1}(\alpha)\right)}{2} d\alpha + \int_{\alpha=k}^{1} \frac{\left(L^{-1}(\alpha) + R^{-1}(\alpha)\right)}{2} d\alpha$$
$$= \int_{\alpha=0}^{k} \frac{\left[\left(a_1 + \left(\frac{\alpha}{k}\right)^{\frac{1}{\xi_1}} (a_2 - a_1)\right) + \left(a_5 - \left(\frac{\alpha}{k}\right)^{\frac{1}{\zeta_2}} (a_5 - a_4)\right)\right]}{2} d\alpha$$
$$+ \int_{\alpha=k}^{1} \frac{\left[\left(a_2 + \left(\frac{\alpha-k}{1-k}\right)^{\frac{1}{\xi_2}} (a_3 - a_2)\right) + \left(a_4 - \left(\frac{\alpha-k}{1-k}\right)^{\frac{1}{\zeta_1}} (a_4 - a_3)\right)\right]}{2} d\alpha$$
$$= \frac{k(a_1 + a_5) + \frac{k(a_2-a_1)}{\left(\frac{1}{\xi_1}+1\right)} - \frac{k(a_5-a_4)}{\left(\frac{1}{\zeta_2}+1\right)} + (1-k)(a_2+a_4) + \frac{(1-k)(a_3-a_2)}{\left(\frac{1}{\xi_2}+1\right)} - \frac{(1-k)(a_4-a_3)}{\left(\frac{1}{\zeta_1}+1\right)}}{2} \tag{31}$$

If we set $\xi_1 = 1$, $\xi_2 = 1$, $\zeta_1 = 1$ and $\zeta_2 = 1$ then the proposed defuzzification technique applies for the linear pentagonal fuzzy number with symmetry as follows

$$\hat{A} = \frac{k(a_1 + a_5) + \frac{k(a_2-a_1)}{2} - \frac{k(a_5-a_4)}{2} + (1-k)(a_2+a_4) + \frac{(1-k)(a_3-a_2)}{2} - \frac{(1-k)(a_4-a_3)}{2}}{2} \tag{32}$$

For $k = 1$ we have

$$\hat{A} = \frac{a_1 + a_2 + a_4 + a_5}{4} \tag{33}$$

4.3|Defuzzification using Removal of Area Method

We consider numerous types of regions of the related linear PFN as illustrated below

$R_1\left(\hat{A}, 0\right)$ = Area of the highlighted region for Figure 13 = $\frac{1}{2}k(a_1 + a_2)$

$R_2\left(\hat{A}, 0\right)$ = Area of the highlighted region for Figure 14 = $\frac{1}{2}(1-k)(a_2 + a_3)$

$R_3\left(\hat{A}, 0\right)$ = Area of the highlighted region for Figure 15 = $\frac{1}{2}(1-k)(a_3 + a_4)$

$R_4\left(\hat{A}, 0\right)$ = Area of the highlighted region for Figure 16 = $\frac{1}{2}k(a_4 + a_5)$

Therefore,

$$\begin{aligned} R\left(\hat{D}, 0\right) &= \frac{R_1\left(\hat{A},0\right) + R_2\left(\hat{A},0\right) + R_3\left(\hat{A},0\right) + R_4\left(\hat{A},0\right)}{4} \\ &= \frac{\frac{1}{2}k(a_1+a_2) + \frac{1}{2}(1-k)(a_2+a_3) + \frac{1}{2}(1-k)(a_3+a_4) + \frac{1}{2}k(a_4+a_5)}{4} \\ &= \frac{k(a_1+a_2) + (1-k)(a_2+a_3) + (1-k)(a_3+a_4) + k(a_4+a_5)}{8}. \end{aligned} \tag{34}$$

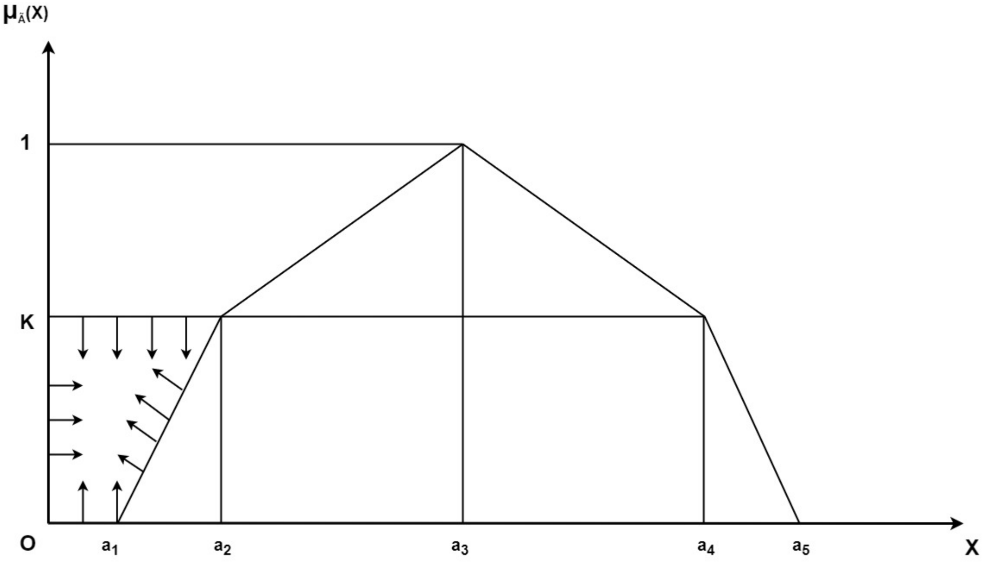

Fig. 13. First step for removal of region method.

For $k = 1$, we have

$$R\left(\hat{D}, 0\right) = \frac{a_1 + a_2 + a_4 + a_5}{8}. \tag{35}$$

4.4|Comparison of the Above Defuzzification Methods

We compared the above proposed defuzzification methods numerically for suitable pentagonal fuzzy number.

Table 1. Numerically Comparison of Defuzzification Methods.

Example	Value of k	Defuzzified Value by Centroid Method	by Mean of α-cut Method	by Removal of Bounded Area Method
$(1, 2, 3, 4, 5)$	1	3	3	1.5
$(1, 2, 3, 4, 5)$	0.5	3	3	1.5
$(-2, -1, 0, 1, 2)$	0.2	0	0	0

5|Numerical Problems

Example 1. *Consider a fuzzy matrix game for two different players A and B whose pay-offs are linear pentagonal fuzzy numbers with symmetry as follows:*

Now using the mean of α- cut method as defuzzification defined by ***Eq. (32)*** for linear pentagonal fuzzy number to convert the given fuzzy pay-off matrix game as

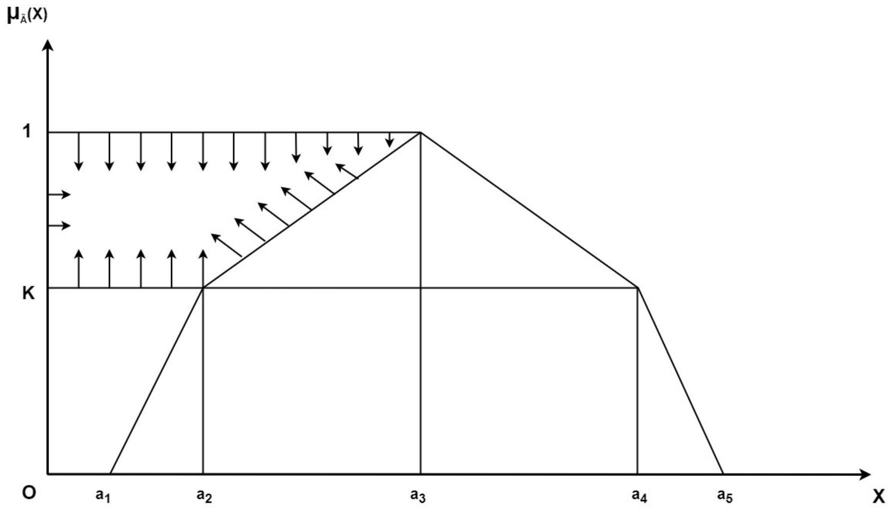

Fig. 14. Second step for removal of region method.

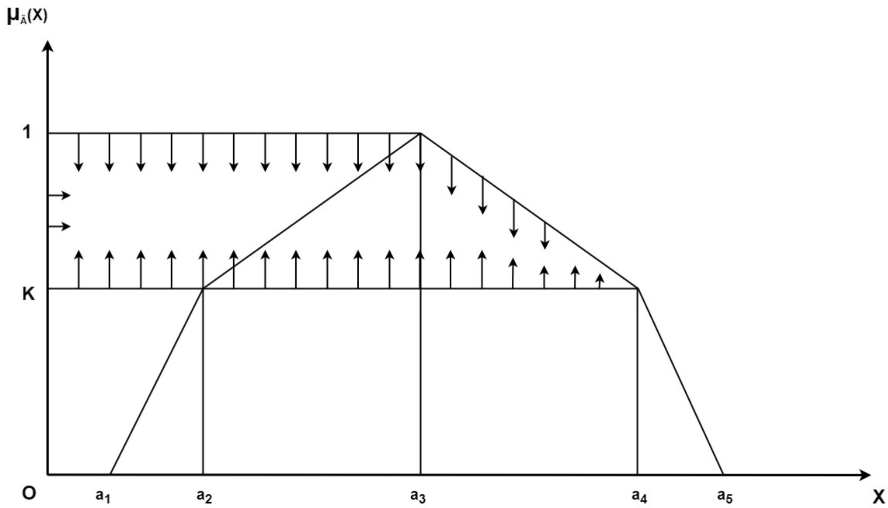

Fig. 15. Third step for removal of region method.

Table (**1**) into the following crisp matrix game as *Table 3*, we have

The above crisp matrix game defined by *Table 3* has no saddle point. Therefore according to the dominance property in matrix games row X_1 can be dominated by row X_3. Hence we have a modified matrix game as *Table 4*.

Again we have that all the elements of column Y_1 are greater than or equal to corresponding elements of column Y_3. Therefore column Y_1 dominates column Y_3 and so we can delete the column Y_1 for new improved matrix game as *Table 5*.

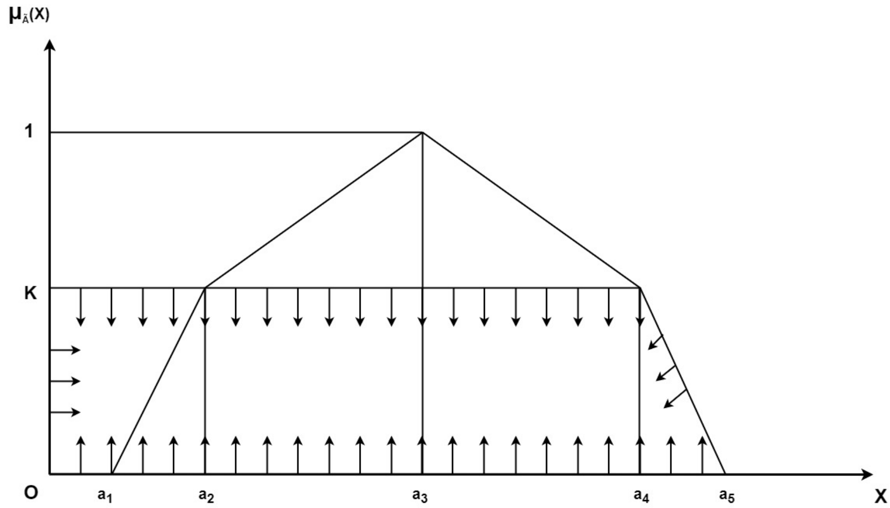

Fig. 16. Fourth step for removal of region method

Table 3. Defuzzified pay-off matrix game.

Strategy	Y_1	Y_2	Y_3	Y_4
X_1	2	1	4	0
X_2	3	4	2	4
X_3	4	2	4	0
X_4	0	4	0	8

Table 4. First step of dominance property.

Strategy	Y_1	Y_2	Y_3	Y_4
X_2	3	4	2	4
X_3	4	2	4	0
X_4	0	4	0	8

Now each element of average of strategies Y_3 and Y_4 of player B are less than or equal to the corresponding elements of strategy Y_2 of player B. Therefore on deleting the column of strategy Y_2 by dominance property, we obtain a new modified matrix game as Table (6).
Again each element of strategy X_2 of player A is less than or equal to the corresponding elements of the average of strategies X_3 and X_4 of player A. Therefore on deleting the row of strategy X_2 by dominance property, we obtain a new improved matrix game as Table (7).
Which is a 2×2 matrix game without saddle point, so the strategy of both the players are $A\left(0, 0, \frac{2}{3}, \frac{1}{3}\right)$ and $B\left(0, 0, \frac{2}{3}, \frac{1}{3}\right)$ respectively and the value of game is $\frac{8}{3}$.

Table 5. Second step of dominance property.

Strategy	Y_2	Y_3	Y_4
X_2	4	2	4
X_3	2	4	0
X_4	4	0	8

Table 6. Third step of dominance property.

Strategy	Y_3	Y_4
X_2	2	4
X_3	4	0
X_4	0	8

Table 7. Fourth step of dominance property.

Strategy	Y_3	Y_4
X_3	4	0
X_4	0	8

Example 2. *Consider a fuzzy matrix game for two different players A and B whose pay-offs are linear pentagonal fuzzy numbers with symmetry as follows:*

Now using the mean of α-cut method as defuzzification defined by equation (32) for the linear pentagonal fuzzy number to convert the given fuzzy pay-off matrix game as Table (8) into the crisp matrix game defined by the Table (9), as follows:

Table 8. Defuzzified pay-off matrix game.

Strategy	Y_1	Y_2	Y_3	Y_4
X_1	3	2	5	0
X_2	4	5	3	5
X_3	5	3	5	0
X_4	0	4	0	9

Since the above crisp matrix game defined by Table (8) has no saddle point, therefore according to the dominance property in matrix games, strategy X_1 of player A can be dominated by strategy X_3 of player A. Hence we have a modified matrix game defined by Table (9).

Table 9. First step of dominance property.

Strategy	Y_1	Y_2	Y_3	Y_4
X_2	4	5	3	5
X_3	5	3	5	0
X_4	0	4	0	9

Again we have that all the elements of strategy Y_1 are greater than or equal to the corresponding elements of strategy Y_3 for player B. Therefore strategy Y_1 dominates strategy Y_3 and so we can delete the strategy column Y_1 for new improved matrix game as Table (10).

Table 10 Second step of dominance property.

Strategy	Y_2	Y_3	Y_4
X_2	5	3	5
X_3	3	5	0
X_4	4	0	9

Further the matrix game as Table (10) can not be reduced by dominance rule. Therefore by applying the linear programming approach for matrix game, we have
For Player A

$$\begin{cases} \max v \\ \text{such that} \\ 5x_1 + 3x_2 + 4x_3 \geq v \\ 3x_1 + 5x_2 + 0x_3 \geq v \\ 5x_1 + 0x_2 + 9x_3 \geq v \\ x_1 + x_2 + x_3 = 1 \\ \text{and} \quad x_1, x_2, x_3, v \geq 0. \end{cases}$$

and
For Player B

$$\begin{cases} \min w \\ \text{such that} \\ 5y_1 + 3y_2 + 5y_3 \leq w \\ 3y_1 + 5y_2 + 0y_3 \leq w \\ 4y_1 + 0y_2 + 9y_3 \leq w \\ y_1 + y_2 + y_3 = 1 \\ \text{and} \quad y_1, y_2, y_3, w \geq 0. \end{cases}$$

Where v, w are the minimum expected gain and maximum expected loss for player A and B respectively. The above linear programming problems for player A and player B can be written in the following manner by assuming $\frac{x_i}{v} = R_i$, $\frac{y_j}{w} = S_j$ for $i, j = 1, 2, 3$ as
For Player A

$$\begin{cases} \min R_1 + R_2 + R_3 \\ \text{such that} \\ 5R_1 + 3R_2 + 4R_3 \geq 1 \\ 3R_1 + 5R_2 + 0R_3 \geq 1 \\ 5R_1 + 0R_2 + 9R_3 \geq 1 \\ \text{and} \quad R_1, R_2, R_3 \geq 0. \end{cases}$$

and
For Player B

$$\begin{cases} \max S_1 + S_2 + S_3 \\ \text{such that} \\ 5S_1 + 3S_2 + 5S_3 \leq 1 \\ 3S_1 + 5S_2 + 0S_3 \leq 1 \\ 4S_1 + 0S_2 + 9S_3 \leq 1 \\ \text{and} \quad S_1, S_2, S_3 \geq 0. \end{cases}$$

On solving the above linear programming problems by the classical simplex method, we obtain $R_1 = 0.2$; $R_2 = 0.08$; $R_3 = 0$; $S_1 = 0$; $S_2 = 0.2$; $S_3 = 0.08$; $\frac{1}{v} = 0.28$ and $\frac{1}{w} = 0.28$. So the optimal strategy of both the players A and B are $A\,(0, 0.71429, 0.28571, 0)$ and $B\,(0, 0, 0.71429, 0.28571)$ respectively and the value of game is 3.57143. In [6] the optimal strategies of the players $A\left(0, 0, \frac{9}{14}, \frac{5}{14}\right)$ and $B\left(0, 0, \frac{9}{14}, \frac{5}{14}\right)$ and the value of game $\frac{45}{14}$ were evaluated.

Example 3. *Consider a fuzzy matrix game for two different players A and B whose pay-offs are linear pentagonal fuzzy numbers with symmetry as follows:*

Now using the mean of α- cut method as defuzzification defined by ***Eq. (32)*** for the linear pentagonal fuzzy number to convert the given fuzzy pay-off matrix game as Table (11) into the crisp matrix game defined by the Table (13), as follows:

Table 11. Defuzzified pay-off matrix game.

Strategy	Y_1	Y_2	Y_3	Y_4
X_1	4	4	6	0
X_2	5	6	5	6
X_3	6	4	6	0
X_4	0	6	0	10

Table 12. Defuzzified pay-off matrix game obtained by [6].

Strategy	Y_1	Y_2	Y_3	Y_4
X_1	4	3	6	0
X_2	5	6	4	6
X_3	6	4	6	0
X_4	0	5	0	10

Since the above crisp matrix game defined by Table (11) has no saddle point, therefore according to the dominance property in matrix games, strategy X_1 can be dominated by strategy X_3 of player A. Hence we have a modified matrix game defined by Table (13).

Table 13. First step of dominance property.

Strategy	Y_1	Y_2	Y_3	Y_4
X_2	5	6	5	6
X_3	6	4	6	0
X_4	0	6	0	10

Again we have that all the elements of strategy Y_1 are equal to the corresponding elements of strategy Y_3 for player B. Therefore either strategy Y_1 or strategy Y_3 for player B can be dominated by each other. Here we delete the strategy Y_1 for the new, improved matrix game as Table (14).

Now each element of the average of strategies Y_3 and Y_4 of player B are less than or equal to the corresponding elements of strategy Y_2 of player B. Therefore on deleting the column strategy Y_2 by dominance property, we obtain a newly modified matrix game defined by the following Table (15) as

Table 14. Second step of dominance property.

Strategy	Y_2	Y_3	Y_4
X_2	6	5	6
X_3	4	6	0
X_4	6	0	10

Table 15. Third step of dominance property.

Strategy	Y_3	Y_4
X_2	5	6
X_3	6	0
X_4	0	10

Further the matrix game as Table (14) can not be reduced by dominance rule. Therefore by applying the graphical approach for $m \times 2$ matrix game according as Table (15) given by Figure 17.
Hence it is clear that the lowest boundary point P on the bounded region will give the smallest expected loss to the minimizing player ($playerB$). So the best strategy for player A are those which pass through the point P. Therefore we can write the matrix game as given by the Table (15) in a 2×2 matrix game defined by the Table (16) as follows:

Table 16. Fourth step of dominance using graphical method.

Strategy	Y_3	Y_4
X_2	5	6
X_3	6	0

The above 2×2 matrix game is without saddle point, so the strategy of both the players are $A\left(0, \frac{6}{7}, \frac{1}{7}, 0\right)$ and $B\left(0, 0, \frac{6}{7}, \frac{1}{7}\right)$ respectively and the value of game is $\frac{36}{7}$. In [6] the optimal strategies of the players $A\left(0, 0, \frac{5}{8}, \frac{3}{8}\right)$, $B\left(0, 0, \frac{5}{8}, \frac{3}{8}\right)$ and the value of the game $\frac{15}{4}$ were evaluated.
Remark 1. *If we delete the strategy Y_3 for new improved matrix game as second step of dominance property and further on going in a similar manner, we obtain the best strategy for player A and B as $A\left(0, \frac{6}{7}, \frac{1}{7}, 0\right)$ and $B\left(\frac{6}{7}, 0, 0, \frac{1}{7}\right)$ respectively and the value of game is $\frac{36}{7}$.*

5.1|Comparative Study of Proposed Defuzzification Techniques by Numerical Examples

In this section we compared all the proposed defuzzification techniques numerically for matrix game problems assigned by [6] and investigate that

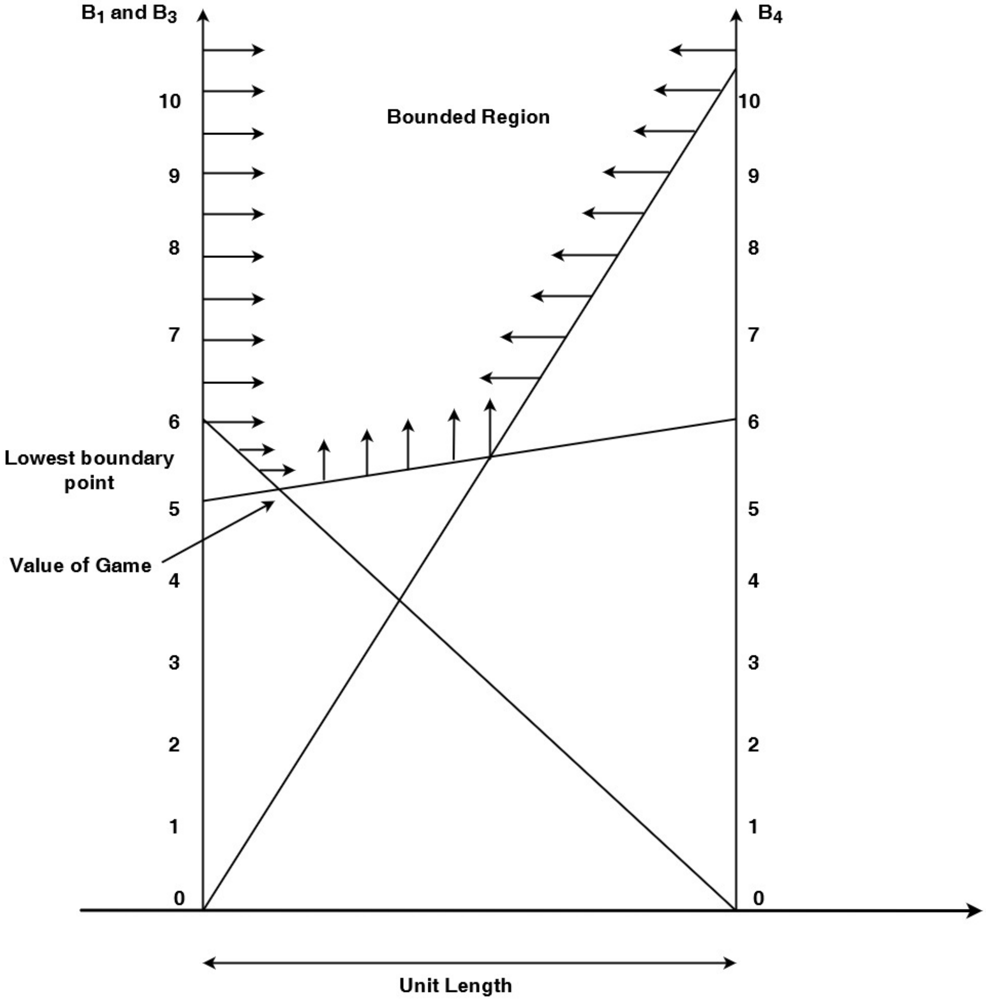

Fig. 17. Graphical representation for third step of dominance property.

I. The strategy of the game players remains unchanged in realistic game problems.
II. Centroid method and mean of α-cut method provides the same results.
III. Value of the matrix game obtained by removal of region method is the half of the game value obtained by other remaining proposed methods.

6|Conclusion

This research paper has systematically explored and elucidated a range of mathematical concepts central to the pentagonal fuzzy numbers. The investigation developed various concepts, including distinct representations, intrinsic properties, α-cuts, and defuzzification methodologies. The application of both linear and non-linear membership functions in a symmetric fashion has been rigorously examined, contributing to a comprehensive understanding of the subject. The incorporation of numerical examples drawn from matrix game problems, as originally provided by [6], has added depth and context to the discourse. The introduction and analysis of novel defuzzification techniques for pentagonal fuzzy numbers have provided valuable insights into optimal strategies for players, as well as the determination of game values—parameters of substantial significance within real-world gaming scenarios. This paper offers a stepping stone for further scholarly inquiries into the field of pentagonal fuzzy numbers across diverse real-life problem domains. By shedding light on the representation, defuzzification, and their applications within the context of game theory, this research develops the way for continued exploration and innovative utilization of pentagonal fuzzy numbers in addressing complex phenomena. Researchers are encouraged to build upon this foundation, unraveling the potential of pentagonal fuzzy numbers as a powerful tool to handle the real-world challenges.

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Data Availability

The data used and analyzed during the current study are available from the corresponding author upon reasonable request.

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